Higher-Order Gauss-Bonnet Cosmology

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We study cosmological models derived from higher-order Gauss-Bonnet gravity $F(R,G)$ by using the Lagrange multiplier approach without assuming the presence of additional fields with the exception of standard perfect fluid matter. The presence of Lagrange multipliers reduces the number of allowed solutions. We need to introduce compatibility conditions of the FRW equations, which impose strict restrictions on the metric or require the introduction of additional exotic matter. Several classes of $F(R,G)$ models are generated and discussed.

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I. INTRODUCTION

Astrophysical data indicate that the observed universe is in an accelerated phase [1]. This acceleration could be induced by the so-called dark energy (see Ref. [2] for a recent review and references therein) the nature and properties of which are not yet understood at fundamental level.

There are several models for dark energy, but, in general, such a constituent can be figured out, in a coarse grain approach, as being constituted by some ideal fluid whose equation of state (EoS) exhibits non-standard properties, in particular, a non standard adiabatic index. On the other hand, dark energy can be considered as a global phenomenon associated with modifications of gravity [3], which are given by extensions of General Relativity (implying modifications of the Hilbert–Einstein action by introducing scalar fields, curvature invariants like the Ricci scalar R or the Ricci and Riemann tensors $R_{\mu\nu}$ and $R^{\alpha}_{\beta\mu\nu}$ or the Gauss–Bonnet topological invariant [4, 5]). Such extended actions describe effective theories coming from fundamental interactions [6] and, from a cosmological point of view, they should consistently describe the early-time inflation and late-time acceleration, without the introduction of any other material dark component. In this picture, the dark side of the universe could be traced as the lack of a final theory of gravity where gravitational interaction does not behave in the same way at all scales [7].

In fact, General Relativity presents problems at ultra-violet (Quantum Gravity) and infra-red (Cosmology) scales [6, 8]. A similar situation appears for dark matter phenomena. No fundamental candidate has been revealed up to now and dynamics of self-gravitating structures could be addressed by modifications of gravity like $f(R)$-gravity (see for example [9]). A further problem is that it is not clear why dark energy had no effect at early epochs while it gives dominant contributions in today observed universe. According to the latest observational data, dark energy currently accounts for about $68 \div 70$ % of the total mass-energy amount of the universe (see, for example, the very recent data coming from the PLANCK collaboration [10]). The main feature of dark energy is that its EoS parameter w_D is negative:

$$w_D = p_D/\rho_D < 0, \tag{1}$$

where ρ_D is the dark energy density and p_D the pressure. Due to this feature, the Hubble fluid is accelerating instead of decelerating as it should be if governed by ordinary perfect fluid matter. Although current data favor the standard ΛCDM cosmology, the uncertainties in the determination of the EoS dark energy parameter w are still too large, namely $w = -1.04^{+0.09}_{-0.10}$. Hence, one is not able to determine, without doubt, which of the three cases: $w < -1$, $w = -1$, or $w > -1$ is the one actually realized in our universe [11, 12]. Furthermore, the evolution of the cosmological constant is a very likely feature in order to connect the early quantum gravity regime with today observed universe [6].

Recently, a new cosmological model describing the different stages of evolution of the universe has been proposed [13]. This model, named Dust of Dark energy, describes the accelerated expansion of the Universe by two scalar fields where one of them is a Lagrange multiplier (LM) which imposes a constraint on the dynamics [14–16]. The extension

to $f(R)$ gravity via the addition of a Lagrange multiplier constraint has been proposed in Ref. [14]. Such a model can be considered as a new version of modified gravity because dynamics and cosmological solutions are different from the standard version of $f(R)$ gravity without such a constraint. This result is clear from a dynamical viewpoint: LMs are anholonomic constraints capable of reducing dynamics [17, 18]. Furthermore, using the LM approach helps in the formulation of covariant renormalizable gravity [19] and can be re-conducted to the existence of Noether symmetries into dynamics [8, 16].

In the present paper, we study the Gauss-Bonnet gravity with LM constraints in view to recover realistic dark energy behaviors. Here we do not specify the form of the gravitational action but generically require a function of the Ricci scalar R and the Gauss-Bonnet invariant G, by considering a generic $F(R,G)$-gravity. In this way, due to the form of the Gauss-Bonnet invariant, all the curvature invariants are taken into account without arbitrary choices. Unlike the models dealt in [16], where the introduction of LM helps to generate a number of new solutions, we found the opposite effect in the case under consideration. Namely, we see that the appearance of LMs reduces the number of solutions and does not help in their generation. This feature is strictly related to the indetermination in the $F(R,G)$ function endowed with all the gravitational degrees of freedom apart from the standard one in the Hilbert-Einstein action R. The paper is organized as follows. In Sect. II, we discuss the main features of $F(R,G)$-gravity where a LM is considered for the Gauss-Bonnet component. In view of studying dark energy models, the cosmological equations are derived. In particular, we recast the further gravitational degrees of freedom in the form of an effective fluid in order to derive a suitable EoS. Sect.III is devoted to the search for cosmological power law solutions for some classes of the $F(R,G)$ function. Results are discussed in Sect.IV where we draw also our conclusions.

II. $F(R,G)$ GRAVITY WITH GAUSS-BONNET LAGRANGE MULTIPLIER

As discussed in [3, 4, 16], higher-order and non local curvature invariants come out in any effective theory derived from unified approaches like strings, braneworld, etc. Here, we will consider a generic $F(R,G)$ function of the Ricci scalar R and the Gauss-Bonnet invariant G, where we ask for a Lagrange multiplier for the Gauss-Bonnet term in order to constrain cosmology. The starting action has the following form:

$$S=\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F(R,G)+\lambda\left(\frac{1}{2}\nabla_\mu G\nabla^\mu G+F_2(G)\right)\right\}+S_m\,. \tag{2}$$

Here $1/2\kappa^2$ is the gravitational coupling constant, $F(R,G)$ and $F_2(G)$ are arbitrary functions, S_m is the standard perfect flud matter action, λ is a Lagrange multiplier whose variation yields the constraint equation

$$\frac{1}{2}\nabla_\mu G\nabla^\mu G+F_2(G)=0\,. \tag{3}$$

G is the Gauss-Bonnet invariant defined as

$$G=R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\mu\nu}R^{\mu\nu}+R^2\,. \tag{4}$$

The gravitational field equations are

$$\begin{aligned}&-\tfrac{1}{2}Fg_{\mu\nu}+\tfrac{\lambda}{2}\nabla_\mu G\nabla_\nu G+F'_RR_{\mu\nu}-\nabla_\mu\nabla_\nu F'_R+g_{\mu\nu}\Box F'_R+2FL'_G\left(R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}+2R_{\mu\alpha\beta\nu}R^{\alpha\beta}-\right.\\&\left.-2R_{\alpha\mu}R^\alpha{}_\nu+RR_{\mu\nu}\right)-4\left(R_{\mu\alpha\beta\nu}+R_{\mu\nu}g_{\alpha\beta}+R_{\alpha\beta}g_{\mu\nu}-R_{\alpha\mu}g_{\beta\nu}-R_{\alpha\nu}g_{\beta\mu}+\right.\\&\left.+\tfrac{1}{2}R\left(g_{\alpha\mu}g_{\beta\nu}-g_{\mu\nu}g_{\alpha\beta}\right)\right)\nabla^\alpha\nabla^\beta FL'_G=\kappa^2T_{\mu\nu}.\end{aligned} \tag{5}$$

Here $F'_R=dF(R,G)/dR$ and $FL'_G=dF(R,G)/dG+\lambda dF_2(G)/dG-\nabla^\mu(\lambda\nabla_\mu G)$. In the 4-dimensional space-time, we have the following expression

$$2\left(R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}+2R_{\mu\alpha\beta\nu}R^{\alpha\beta}-2R_{\alpha\mu}R^\alpha{}_\nu+RR_{\mu\nu}\right)-\frac{1}{2}Gg_{\mu\nu}=0.$$

Hence Eq. (5) can be rewritten as

$$\begin{aligned}&-\tfrac{1}{2}Fg_{\mu\nu}+\tfrac{\lambda}{2}\nabla_\mu G\nabla_\nu G+F'_RR_{\mu\nu}-\nabla_\mu\nabla_\nu F'_R+g_{\mu\nu}\Box F'_R+\tfrac{1}{2}FL'_GGg_{\mu\nu}-4\left(R_{\mu\alpha\beta\nu}+R_{\mu\nu}g_{\alpha\beta}+\right.\\&\left.+R_{\alpha\beta}g_{\mu\nu}-R_{\alpha\mu}g_{\beta\nu}-R_{\alpha\nu}g_{\beta\mu}+\tfrac{1}{2}R\left(g_{\alpha\mu}g_{\beta\nu}-g_{\mu\nu}g_{\alpha\beta}\right)\right)\nabla^\alpha\nabla^\beta FL'_G=\kappa^2T_{\mu\nu}\,.\end{aligned} \tag{6}$$

Let us now consider a Friedmann-Robertson-Walker (FRW) universe with the flat spatial metric

$$ds^2=-dt^2+a(t)^2\sum_{i=1,2,3}\left(dx^i\right)^2\,. \tag{7}$$

Then from Eq. (3) one obtains

$$F_2(G) = \frac{\dot{G}^2}{2} \,. \tag{8}$$

FRW equations are specified as soon as one defines the effective density and pressure

$$\rho_{\rm eff} \equiv \frac{1}{F'_R}\left\{\rho_{\rm M} + \frac{1}{2\kappa^2}\left[(F'_R R + G\, FL'_G - F) - \lambda\dot{G}^2 - 6H\dot{F'}_R - 24H^3\dot{FL'}_G\right]\right\}\,, \tag{9}$$

$$p_{\rm eff} \equiv \frac{1}{F'_R}\left\{p_{\rm M} + \frac{1}{2\kappa^2}\left[-\left(F'_R R + G\, FL'_G - F\right) + 4H\dot{F'}_R + 2\ddot{F'}_R \right.\right.$$
$$\left.\left. + 16H\left(\dot{H} + H^2\right)\dot{FL'}_G + 8H^2\ddot{FL'}_G\right]\right\}\,. \tag{10}$$

Here $\rho_{\rm eff} = 3\kappa^{-2}H^2$ and $p_{\rm eff} = -\kappa^{-2}\left(2\dot{H} + 3H^2\right)$, respectively. A general method for solving such equations does not exist. However, a few solutions for general $F(R,G)$ theory have been found [20–24]. We can reduce the problem, if we assume that

$$FL'_G = F'_G, \;\; \lambda\dot{G}^2 = 2\kappa^2\rho_{\rm M}, \;\; p_{\rm M} = 0. \tag{11}$$

We obtain an equation for the Lagrange multiplier, which will lead us to a model where the material field is in the form of dust ($p = 0$). In such a case the conservation equation for matter

$$\dot{\rho_{\rm M}} + 3H(\rho_{\rm M} + p_{\rm M}) = 0\,, \tag{12}$$

could result violated but the general Bianchi identities have to remain conserved. Let us consider in more detail Eqs. (9)-(10). It is easy to remove from these equations the constraint derived from the Lagrangian multiplier. We obtain then the compatibility condition of the equations, which takes the form

$$\begin{aligned} & 3H\kappa^2 p_{\rm M} + 3H\kappa^2\rho_{\rm M} + \kappa^2\dot{\rho_{\rm M}} + 288H\lambda\left[2\dot{H}\left(2H^2 + \dot{H}\right) + H\ddot{H}\right] \times \\ \times \quad & \left[2\left(\dot{H}^2\left(-6H^2 + 96H^6 + \dot{H}(-1 + 96H^4 + 24H^2\dot{H})\right) + \right.\right. \\ + \quad & \left.\left. H\left(-2H^2 + 3\dot{H}(-1 + 16H^4 + 8H^2\dot{H})\right)\ddot{H} + 6H^4\ddot{H}^2\right) - H^2\dddot{H}\right] = 0. \end{aligned} \tag{13}$$

This equation can be rewritten in terms of the Gauss-Bonnet invariant obtaining

$$\lambda\dot{G}\left(\dot{G}^2 - 2\ddot{G}\right) = -4\kappa^2\left(3H(p_{\rm M} + \rho_{\rm M}) + \dot{\rho_{\rm M}}\right). \tag{14}$$

If the matter satisfies the conservation law we need to impose the additional conditions $G = \text{const}$ or $G = c_1 + 2\ln(t + c_2)$.

A similar situation arises when we consider another Lagrangian of slightly different type, that is

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F(R,G) + \lambda\left(\frac{1}{2}\nabla_\mu R\nabla^\mu R + F_2(R)\right)\right\} + S_m\,, \tag{15}$$

where λ is again a Lagrange multiplier and F_2 is an arbitrary function of the scalar curvature R.

In this case the compatibility condition of the equations of motion will have exactly the same form, but with the replacement of the invariant of the Gauss-Bonnet with the Ricci curvature scalar

$$\lambda\dot{R}\left(\dot{R}^2 - 2\ddot{R}\right) = -4\kappa^2\left(3H(p_{\rm M} + \rho_{\rm M}) + \dot{\rho_{\rm M}}\right). \tag{16}$$

Clearly Eqs. (14) and (16) do not contradict the validity of the general Bianchi identities for the theory but point out that (Gauss-Bonnet) curvature terms, selected by the Lagrange multipliers, can act as source terms. With these considerations in mind, let us search for cosmological solutions for $F(R,G)$ gravity according to the Lagrange multiplier method.

III. COSMOLOGICAL SOLUTIONS FOR $F(R,G)$ GRAVITY

Our aim is now to search for cosmological solutions compatible with $F(R,G)$ gravity that, eventually, select the form of the function. Let us start with power law solutions where the scale factor behaves as

$$a(t) = a_0 t^{h_0}, \tag{17}$$

so that the Hubble rate is given by

$$H = \frac{h_0}{t}. \tag{18}$$

The FRW equations assume the form

$$\begin{aligned} 0 &= (F - 2\kappa^2\rho_{\mathrm{M}})t^{14} - 24(h_0-1){h_0}^3 t^{10} F(R,G)'_G - 6(h_0-1)h_0 t^{12} F(R,G)'_R + \\ &+ 6h_0\left(-384(h_0-1){h_0}^5\left(48(h_0-1){h_0}^3(9+h_0) + (29 - h_0(14+3h_0))t^4\right)\lambda + \right. \\ &+ \left.\left(4{h_0}^2 t^{10}\dot{F(R,G)}'_G + t^{12}\dot{F(R,G)}'_R + 384(h_0-1){h_0}^5\left(\left(48(h_0-1){h_0}^3 + (9-2h_0)t^4\right)\dot{\lambda} - t^5\ddot{\lambda}\right)\right)\right), \end{aligned} \tag{19}$$

and

$$\begin{aligned} 0 &= (F + 2\kappa^2 p_{\mathrm{M}})t^{14} - 24(h_0-1){h_0}^3 t^{10} F(R,G)'_G + 2(1-3h_0)h_0 t^{12} F(R,G)'_R + \\ &+ 2\left(-1152(h_0-1){h_0}^5\left(16(h_0-1){h_0}^3(10+h_0)(3h_0-13) - (h_0-3)(6+h_0)(3h_0-5)t^4\right)\lambda + \right. \\ &+ t\left(8(h_0-1){h_0}^2 t^{10}\dot{F(R,G)}'_G + 2h_0 t^{12}\dot{F(R,G)}'_R - 405504{h_0}^8\dot{\lambda} + 847872{h_0}^9\dot{\lambda} - 479232{h_0}^{10}\dot{\lambda} + 36864{h_0}^{11}\dot{\lambda} + \right. \\ &+ 42240{h_0}^5 t^4\dot{\lambda} - 64896{h_0}^6 t^4\dot{\lambda} + 23808{h_0}^7 t^4\dot{\lambda} - 1152{h_0}^8 t^4\dot{\lambda} + 4{h_0}^2 t^{11}\ddot{F(R,G)}'_G + t^{13}\ddot{F(R,G)}'_R + 18432{h_0}^8 t\ddot{\lambda} - \\ &- \left.\left. 36864{h_0}^9 t\ddot{\lambda} + 18432{h_0}^{10} t\ddot{\lambda} - 6528{h_0}^5 t^5\ddot{\lambda} + 8448{h_0}^6 t^5\ddot{\lambda} - 1920{h_0}^7 t^5\ddot{\lambda} - 384(h_0-1){h_0}^5 t^6\lambda^{(3)}\right)\right). \end{aligned} \tag{20}$$

If we choose the Lagrange multiplier as

$$\lambda = c\, e^{-\frac{12(h_0-1){h_0}^3}{t^4}} t^{5-3h_0}, \tag{21}$$

were c is a constant, matter behaves as "dust" assuming the form

$$\rho_{\mathrm{M}} = \frac{9216c}{2\kappa^2} e^{-\frac{12(h_0-1){h_0}^3}{t^4}} (h_0-1)^2 {h_0}^6 t^{-5-3h_0}, \quad p_{\mathrm{M}} = 0\,, \tag{22}$$

where the pressure is clearly zero.

Choosing, instead, a Lagrange multiplier of the form

$$\lambda = \lambda_1 t^{\lambda_2}\,, \tag{23}$$

one can use the reconstruction method proposed in Refs. [16, 20–22]. However, in this case we need to introduce the matter in the special form

$$\begin{aligned} \rho_{\mathrm{M}} &= c_1 t^{-3h_0(1+w)} + \frac{4608(h_0-1)^2 h_0^6 \lambda_1 t^{-14+\lambda_2}}{\kappa^2}\left(\frac{48(h_0-1){h_0}^3}{-14+\lambda_2+3h_0(1+w)} - \frac{5t^4}{-10+\lambda_2+3h_0(1+w)}\right), && (24) \\ p_{\mathrm{M}} &= w\rho_{\mathrm{M}}. && (25) \end{aligned}$$

If $\lambda_2 = 14 - 3h_0(1+w)$ then the matter density takes the form

$$\rho_{\mathrm{M}} = t^{-3h_0(1+w)}\left(c_1 + \frac{1152(h_0-1)^2 h_0^6 \lambda_1\left(-5t^4 + 192(h_0-1)h_0^3 \ln t\right)}{\kappa^2}\right). \tag{26}$$

On the other hand, if $\lambda_2 = 10 - 3h_0(1+w)$, the matter density takes the form

$$\rho_{\mathrm{M}} = t^{-3h_0(1+w)}\left(c_1 + \frac{4608(h_0-1)^2 h_0^6 \lambda_1\left(-\frac{12(h_0-1)h_0^3}{t^4} - 5\ln t\right)}{\kappa^2}\right). \tag{27}$$

With proper functions $P(\phi)$, $Z(\phi)$ and $Q(\phi)$ of a scalar field ϕ, which we can identify with the time t, we represent the term $\mathcal{F}(R,G)$ in the action in Eq. (2) as $P(t)R + Z(t)G + Q(t)$. Varying this action with respect to t, we find $(dP(t)/dt)\,R + (dZ(t)/dt)\,G + dQ(t)/dt = 0$. By solving this equation, we obtain $t = t(R,G)$. Combining this and using the above representation, we get the special case $\mathcal{F}(R,G) = P(t)R + Z(t)G + Q(t)$.

Let us consider now a few cases that can be worked out in detail.

A. The case of $F(G)$ gravity with a Lagrange multiplier

Let us start with a function the form $F = R + F(G)$, where we consider a pure Gauss-Bonnet function as correction to Einstein gravity. Considering $c_1 = 0$ (without ordinary matter) then it is easy to find an explicit form for $F(G)$. There are two cases

1. We have

$$\begin{aligned}
F'_G &= \frac{4(h_0-1)h_0^2(3h_0-2)\left(-10+h_0+3h_0^2\right)}{(h_0-2)^2(5+h_0)\left(-1+2h_0+3h_0^2\right)t^2},\\
\lambda &= \frac{(2-3h_0)t^8}{1152h_0^4\left(-2+7h_0-h_0^2-7h_0^3+3h_0^4\right)},\\
t &\to \frac{2^{3/4}3^{1/4}\left(-h_0^3+h_0^4\right)^{1/4}}{G^{1/4}},\\
F(G) &= \frac{2\sqrt{\frac{2}{3}}G^{3/2}\sqrt{(h_0-1)h_0^3}\left(20-32h_0-3h_0^2+9h_0^3\right)}{3(h_0-2)^2h_0(5+h_0)\left(-1+2h_0+3h_0^2\right)},
\end{aligned} \tag{28}$$

where the Gauss-Bonnet function scales as a 3/2 power. The dark energy behavior means that the power h_0 in Eq.(17) is larger than 1.

2. Another case is

$$\begin{aligned}
F'_G &= \frac{\left(84-20h_0-143\mathrm{h0}^2-30h_0^3+9h_0^4\right)t^6}{192(-3+h_0)(-1+h_0)h_0^4(2+3h_0)\left(2-7h_0+3h_0^2\right)},\\
\lambda &= \frac{(-2+3h_0)t^{12}}{18432(-1+h_0)^2h_0^7\left(2-7h_0+3h_0^2\right)},\\
F &= \frac{\sqrt{\frac{3}{2}}\sqrt{(-1+h_0)h_0^3}\left(-84+20h_0+143h_0^2+30h_0^3-9h_0^4\right)}{\sqrt{G}(-3+h_0)h_0(2+3h_0)\left(2-7h_0+3h_0^2\right)}.
\end{aligned} \tag{29}$$

If we eliminate the Lagrange multiplier we get the following form of the function $F(G)$

$$\begin{aligned}
F'_G &= g_1 t^{11/3},\\
F(G) &= \frac{32}{27}g_1\left(\sqrt{2}\,3^{1/4}+11G^{1/4}\right)G^{1/12}
\end{aligned} \tag{30}$$

and then we get another fractional power for $F(G)$. It is worth noticing that this kind of fractional behavior for generic gravitational actions is usually determined by the presence of Noether symmetries for the dynamical system (see [8] for details). This shows the strict relation between LM method and the Noether Symmetry Approach to reduce dynamical systems.

B. The case of $F(R,G)$ gravity in presence of ordinary matter

Let us assume now a generic $F(R,G)$ function with no Lagrange multiplier, but the presence of matter in the form of a standard perfect fluid like $\rho_{\mathrm{M}} = \rho_0 a^{-3(1+w)}$, $p_{\mathrm{M}} = w\rho_0 a^{-3(1+w)}$. The FRW Eqs. (9-10) take the following form

$$-2\kappa^2\rho_0 t^4 + a_0^3 t^{3h_0}(a_0 t^{h_0})^{3w}(Ft^4 - 6h_0(4g_1h_0^2(-1-g_2+h_0)t^{f_2} + r_1(-1+h_0-r_2)t^{2+r_2})) = 0, \tag{31}$$

$$\begin{aligned}
{a_0}^3 t^{3h_0}(a_0t^{h_0})^{3w}\left(-Ft^4 - 8g_1(g_2-h_0)h_0^2(-3+f_2+3h_0)t^{f_2} + 2r_1(h_0-r_2)(-1+3h_0+r_2)t^{2+r_2}\right) &- \\
-2\kappa^2\rho_0t^4w &= 0.
\end{aligned} \tag{32}$$

Here, the functions F'_R and F'_G can be selected in the form $F'_R = r_1 t^{r_2}$ and $F'_G = g_1 t^{g_2}$. Using the reconstruction method [20–22] which we described above (that is $Z(t) \equiv F'_G$, $P(t) \equiv F'_R$ and $Q(t) = F(R,G) - F'_R R - F'_G G$), we can find t as a function of R and G, and then F as a function of R and G. However, this will lead to an equation containing an irrational degree for t, which cannot be solved in general. As above, one can consider a few special cases, but just for fixed values of h_0 and w. In fact, deriving the explicit form of F as a function of R and G is very difficult. We choose, for the sake of simplicity, the case when $F(R,G) = F_1(R) + F_2(G)$. Then the solution is possible, for $\rho_0 \neq 0$, if

1. $g_2 = 4 - 3h_0(1+w) \neq 2 + r_2$,
2. $2 + r_2 = 4 - 3h_0(1+w) \neq g_2$,
3. $g_2 = 2 + r_2 = 4 - 3h_0(1+w)$.

Let us consider the first case. Solving Eqs. (31)-(32) we find

$$\rho_0 = -4a_0^{3+3w} g_1 h_0^2 (-4 + 3h_0(1+w)(-1 + h_0(4+3w))/(\kappa^2(1+w)), \tag{33}$$

$$r_2 = \frac{1}{2}\left(1 + h_0 \pm \sqrt{1 + 10h_0 + h_0^2}\right). \tag{34}$$

It is evident that, in order that the condition $\rho_0 > 0$ be satisfied, we need to impose restrictions on g_1 and h_0, namely: $g_1 > 0$ and $1/4 < h_0 < 4/3$ or $g_1 > 0$ and $h_0 < 1/4$ or $h_0 > 4/3$. Finally the functions $F_1(R)$ and $F_2(G)$ assume the form

$$F_1(R) = \frac{2^{\frac{1}{4}\left(9+h_0\pm\sqrt{1+10h_0+h_0^2}\right)} 3^{\frac{1}{4}\left(1+h_0\pm\sqrt{1+10h_0+h_0^2}\right)} (h_0(2h_0-1))^{\frac{1}{4}\left(1+h_0\pm\sqrt{1+10h_0+h_0^2}\right)} R^{\frac{1}{4}\left(3-h_0\mp\sqrt{1+10h_0+h_0^2}\right)} r_1}{3 - h_0 \mp \sqrt{1+10h_0+h_0^2}} \tag{35}$$

$$F_2(G) = g_1 \frac{2^{5-\frac{9}{4}h_0(1+w)} 3^{-\frac{3}{4}h_0(1+w)} (h_0-1)^{1-\frac{3}{4}h_0(1+w)} h_0^{2-\frac{39}{4}h_0(1+w)}}{1+w} G^{\frac{3}{4}h_0(1+w)}. \tag{36}$$

In the second case the functions $F_{1,2}$ assume the forms

$$F_1(R) = \frac{r_1 2^{2-\frac{3}{2}h_0(1+w)} 3^{-\frac{3}{2}h_0(1+w)} (2h_0-1)(h_0(2h_0-1))^{-\frac{3}{2}h_0(1+w)}}{1+w} R^{\frac{3}{2}h_0(1+w)}, \tag{37}$$

$$F_2(G) = -g_1 2^{2+\frac{3(3+h_0)}{4}} 3^{\frac{3+h_0}{4}} (h_0-1)^{\frac{1}{4}(h_0-1)} h_0^{\frac{3(3+h_0)}{4}} G^{\frac{1}{4}-\frac{h_0}{4}}. \tag{38}$$

If $\rho_0 = 0$ then the function F is the sum of the functions $F_2(G)$ of the first case (36) and the functions $F_1(R)$ of the second case (37).

For the third case we easily find the function F by summing up the functions

$$F_1(R) = r_1 \frac{2^{2-\frac{3}{2}h_0(1+w)} 3^{-\frac{3}{2}h_0(1+w)} (2h_0-1)(h_0(-1+2h_0))^{-\frac{3}{2}h_0(1+w)}}{1+w} R^{\frac{3}{2}h_0(1+w)}, \tag{39}$$

$$F_2(G) = g_1 \frac{2^{5-\frac{9}{4}h_0(1+w)} 3^{-\frac{3}{4}h_0(1+w)} (h_0-1)^{1-\frac{3}{4}h_0(1+w)} h_0^{2-\frac{9}{4}h_0(1+w)}}{1+w} G^{\frac{3}{4}h_0(1+w)}. \tag{40}$$

As general remark, also in these cases the functions F are fractional power laws pointing out that their form could be easily related to the presence of a Noether symmetry for the dynamical system.

IV. DISCUSSION AND CONCLUSIONS

We have examined cosmological models derived from $F(R,G)$ gravity in presence of LMs like in the action (2). No other additional field has been considered with the exception of ordinary perfect fluid matter. Despite of Refs. [15, 16], where with the help of LMs new solutions were generated, here, we have constraints like (14) and (16) that have to be taken into account restricting the possible forms of the function $F(R,G)$. On the other hand we can introduce, in addition to standard matter (in the form of a perfect fluid satisfying the conservation equation), some form of exotic matter, which should compensate, in Eqs. (14) and (16), the terms corresponding to LMs. For a power law behavior of the cosmological scale factor $a(t)$ and the LM of the form (21), we have to introduce matter in the form (22), for which our model reduces to the standard $F(R,G)$ model without LMs.

To generate other solutions one can choose the LM in the form (23); then the matter of the form (27) will provide the compatibility of FRW equations. All of these examples are obtained for power law behaviors of the scale factor. Another example is based on $F(G)$ gravity. In this case it is easy to reconstruct the explicit form of the function $F(R)$. For a LM in the form (2) this function has the form (28) and (29); with the exclusion of the LM, the function $F(G)$ has the form (30). All of these solutions have been built in the absence of ordinary matter, but the addition of such matter does not change much the situation. As examples, we have built several solutions for $F(R)+F(G)$ gravity with matter, but without LMs like (30) -(40). Adding LM, as noted above, leads to the appearance of exotic matter and significant complication in equations: this means a series of additional restrictions on the $F(R,G)$ functions. In conclusion, the introduction of LMs can lead to additional restrictions, which can be avoided in the models with further fields, such as scalar fields. In this case, the compatibility of the FRW equations is achieved by the right choice of a scalar field. Standard choices for it, for example, $\phi = \ln t$, allow to give rise to consistent sets of equations as discussed in [16]. Finally, we can say that the LM method is a sort of inverse scattering approach that allows to reconstruct a model, given the cosmological behavior that one wants to fit. This feature is particularly relevant with respect to the issue of matching models with observational data instead of imposing, a priori, arbitrary theories.

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